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TRANSIENT CONDUCTION IN A TWO-DIMENSIONAL COMPOSITE SLAB—II. PHYSICAL INTERPRETATION OF TEMPERATURE MODES

H. SALT
Division of Energy Technology, Commonwealth Scientific and Industrial Research Organization, Highett, Victoria 3190, Australia

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Abstract—It is shown that the eigenvalues of the temperature modes across a two-layer composite slab must progressively become imaginary in the more diffuse layer, for the higher order temperature modes along the slab. A logical argument is presented to show that this represents the more diffuse layer effectively short-circuiting the less diffuse layer in dissipating temperature variations along its length. It is also shown that for a fully insulated composite slab, temperature variations along the slab must be accompanied by temperature variations across the slab.

NOMENCLATURE

- A_j, B_j, C_j constants in the series expansion of the temperature in the slab
- a_j thermal diffusivity of the jth layer of the slab
- B_t Biot number, hx_J/k_J
- E_2, G_2 defined by equation (17)
- h heat transfer coefficient from the surface of the slab
- J number of layers in the slab
- k_j conductivity of the jth layer of the slab
- T temperature in the slab
- t time
- x position across the slab
- x_J thickness of the slab
- y position along the slab
- y_1 length of the slab

Greek symbols

- α_j normalized thermal diffusivity, a_j/a_J
- β_j normalized thermal conductivity, k_j/k_J
- γ normalized length of the slab, y_1/x_J
- Δ difference between consecutive longitudinal eigenvalues
- η normalized position along the slab, y/x_J
- θ normalized time, ta_J/k_J
- λ transverse eigenvalue
- μ longitudinal eigenvalue
- ψ normalized position across slab, x/x_J

INTRODUCTION

THE TRANSIENT temperature response of a two-dimensional multi-layer composite slab, which experiences a sudden change in its environmental temperature, has been expressed in terms of the product of a doubly infinite series in the two spatial dimensions together with an exponential time dependency [1]. The slab under consideration is fully insulated on three sides and is coupled to its environment only through the upper surface of the uppermost layer of the slab, but since there is no energy transfer across the lower surface of the slab, the solution is equally applicable to a two-dimensional composite slab coupled to its environment through both the upper and lower layers. The terms of the series expansions, or the eigenfunctions, can be thought of as temperature modes within the slab. The study was stimulated by a solar space-heating system and in order to make full use of the results of the study, it is important that a full physical interpretation of the solution be made. The solution has the form

$$T = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \exp\{-(\lambda_{jnm}^2 + \mu_n^2)\theta\} C_{jn} \cos \mu_n \eta \times (A_{jnm} \cos \lambda_{jnm}\psi + B_{jnm} \sin \lambda_{jnm}\psi), \quad (1)$$

where μ_n are the eigenvalues associated with the space coordinate along the slab and λ_{jnm} are those associated with the space coordinate across the slab. Physical interpretations and implications of solution (1) will now be made.

THE LONGITUDINAL EIGENFUNCTIONS

The eigenvalues μ_n associated with the longitudinal space coordinate η have values of $n\pi/\gamma, n = 0, 1, 2, \ldots$, as given by equation (11) in ref. [1], and hence the associated eigenfunctions, which have the form $\cos \mu_n\eta$, are simply the terms of a Fourier cosine spectrum in the η-coordinate of the initial temperature distribution. If this happens to be separable, then the amplitudes of the Fourier components can be determined but otherwise, only the amplitude of the product of the Fourier component and an associated transverse eigenfunction can be determined. The relative rates of decay of these Fourier components is of interest in the space-heating system for which this study was initiated, and this aspect of the differences between the homogeneous and composite slabs will now be considered.

For the homogeneous slab, the magnitude of the pth Fourier component of the initial temperature distribution at the point (0, 0) and time θ, relative to its value of $\theta = 0$, is, from equation (53) in ref. [1]

$$\exp(-\mu_p^2\theta) \sum_{m=1}^{\infty} A_{1m} \exp(-\lambda_{1m}^2\theta) / \sum_{m=1}^{\infty} A_{1m}.$$

The rate of decay of this normalized value for the pth Fourier component relative to that for the qth Fourier component is therefore

$$\exp\{-(\mu_p^2-\mu_q^2)\theta\}.$$

Hence the relative rates of decay of the Fourier components of the initial temperature distribution along the slab, are determined solely by the relative magnitudes of the eigenvalues of the Fourier components and are independent of any temperature redistribution or decay in the ψ-coordinate.

For the composite slab, the magnitude of the pth Fourier component of the initial temperature distribution at the point (0, 0) and time θ, relative to its value at $\theta = 0$, is from equation (24) in ref. [1]

$$\exp(-\mu_p^2\theta) \sum_{m=1}^{\infty} A_{1pm} \exp(-\lambda_{1pm}^2\theta) / \sum_{m=1}^{\infty} A_{1pm}.$$

The temporal rate of decay of this normalized value for the pth Fourier component relative to that for the qth Fourier component is therefore

$$\exp\{-(\mu_p^2-\mu_q^2)\theta\} \frac{\sum_{m=1}^{\infty} A_{1pm} \exp(-\lambda_{1pm}^2\theta) \sum_{m=1}^{\infty} A_{1qm}}{\sum_{m=1}^{\infty} A_{1qm} \exp(-\lambda_{1qm}^2\theta) \sum_{m=1}^{\infty} A_{1pm}}.$$

If the initial temperature distribution is separable then, as was shown in equation (43) in ref. [1], $\sum_{m=1}^{\infty} A_{1nm}$ is a constant for all values of 'n' so that, in this special case, the above expression becomes

$$\exp\{-(\mu_p^2-\mu_q^2)\theta\} \frac{\sum_{m=1}^{\infty} A_{1pm} \exp(-\lambda_{1pm}^2\theta)}{\sum_{m=1}^{\infty} A_{1qm} \exp(-\lambda_{1qm}^2\theta)}.$$

For sufficiently large values of θ, only the first terms of the above two series are relevant and the expression can then be reduced to

$$\exp[-\{(\mu_p^2+\lambda_{1p1}^2)-(\mu_q^2+\lambda_{1q1}^2)\}\theta] A_{1p1}/A_{1q1}.$$

Hence, for the composite slab, the rates of decay of the Fourier components, relative to each other, are not only dependent on the relative magnitudes of these respective eigenvalues but also on the magnitudes of the associated transverse eigenvalues. In principle, this phenomenon would allow one to modify the longitudinal transient behaviour of the composite slab by changing the transverse transient behaviour, and this can be done by altering the relative thicknesses of the layers, their relative physical properties and the coupling between the slab and its surroundings through the free surface.

THE TRANSVERSE EIGENVALUES

The eigenvalues associated with the space coordinate ψ are given by equations (22) and (23) in ref. [1] for the multi-layer slab and by equation (52) for the homogeneous slab. The roots of equation (52) are necessarily real [4], but it is clear from the form of equation (23) that the eigenvalues in some layers of the composite slab could be imaginary. They can never be imaginary in all layers simultaneously since if $\lambda_j = i\lambda_{ij}$, then equation (22) in ref. [1] becomes

$$[B_t \quad 1]\begin{bmatrix}1 & 0\\ 0 & i\beta_J\lambda_{iJ}\end{bmatrix}\Omega[i\lambda_{iJ}(\psi_J-\psi_{J-1})]$$
$$\times\begin{bmatrix}1 & 0\\ 0 & \dfrac{\beta_j\lambda_{ij}}{\beta_{j+1}\lambda_{ij+1}}\end{bmatrix}\Omega[i\lambda_{ij}(\psi_j-\psi_{j-1})]$$
$$\times\begin{bmatrix}1 & 0\\ 0 & \dfrac{\beta_{j-1}\lambda_{ij-1}}{\beta_j\lambda_{ij}}\end{bmatrix}\Omega[i\lambda_{ij-1}(\psi_{j-1}-\psi_{j-2})]\times\cdots$$
$$\times\begin{bmatrix}1 & 0\\ 0 & \dfrac{\beta_1\lambda_{i1}}{\beta_2\lambda_{i2}}\end{bmatrix}\Omega[i\lambda_{i1}\psi_1]\begin{bmatrix}1\\0\end{bmatrix} = 0. \tag{2}$$

Now from the definition of Ω, equation (17) in ref. [1]

$$\Omega[i\lambda_{ij}(\psi_j-\psi_{j-1})] = \begin{bmatrix}\cosh\lambda_{ij}(\psi_j-\psi_{j-1}) & i\sinh\lambda_{ij}(\psi_j-\psi_{j-1})\\ -i\sinh\lambda_{ij}(\psi_j-\psi_{j-1}) & \cosh\lambda_{ij}(\psi_j-\psi_{j-1})\end{bmatrix}, \tag{3}$$

and therefore multiplying any two adjacent general matrix factors in equation (2) gives

$$\begin{bmatrix}1 & 0\\ 0 & \dfrac{\beta_j\lambda_{ij}}{\beta_{j+1}\lambda_{ij+1}}\end{bmatrix}\Omega[i\lambda_{ij}(\psi_j-\psi_{j-1})]$$
$$\times\begin{bmatrix}1 & 0\\ 0 & \dfrac{\beta_{j-1}\lambda_{ij-1}}{\beta_j\lambda_{ij}}\end{bmatrix}\Omega[i\lambda_{ij-1}(\psi_{j-1}-\psi_{j-2})]$$
$$= \begin{bmatrix}S_{11} & iS_{12}\\ -iS_{21} & S_{22}\end{bmatrix}, \tag{4}$$

where

$$S_{11} = \cosh\lambda_{ij}(\psi_j-\psi_{j-1})\cosh\lambda_{ij-1} \times(\psi_{j-1}-\psi_{j-2}) + \frac{\beta_{j-1}\lambda_{ij-1}}{\beta_j\lambda_{ij}}\sinh\lambda_{ij}(\psi_j-\psi_{j-1}) \times\sinh\lambda_{ij-1}(\psi_{j-1}-\psi_{j-2}), \tag{5}$$

$$S_{12} = \cosh\lambda_{ij}(\psi_j-\psi_{j-1})\sinh\lambda_{j-1} \times(\psi_{j-1}-\psi_{j-2}) + \frac{\beta_{j-1}\lambda_{ij-1}}{\beta_i\lambda_{ij}}\sinh\lambda_{ij}(\psi_j-\psi_{j-1}) \times\cosh\lambda_{ij-1}(\psi_{j-1}-\psi_{j-2}), \tag{6}$$

$$S_{21} = \frac{\beta_j\lambda_{ij}}{\beta_{j+1}\lambda_{ij+1}} \sinh \lambda_{ij}(\psi_j - \psi_{j-1})$$
$$\times \cosh \lambda_{ij-1}(\psi_{j-1} - \psi_{j-2}) + \frac{\beta_{j-1}\lambda_{ij-1}}{\beta_{j+1}\lambda_{ij+1}}$$
$$\times \cosh \lambda_{ij}(\psi_j - \psi_{j-1}) \sinh \lambda_{ij-1}(\psi_{j-1} - \psi_{j-2}), \quad (7)$$
$$S_{22} = \frac{\beta_j\lambda_{ij}}{\beta_{j+1}\lambda_{ij+1}} \sinh \lambda_{ij}(\psi_j - \psi_{j-1})$$
$$\times \sinh \lambda_{ij-1}(\psi_{j-1} - \psi_{j-2}) + \frac{\beta_{j-1}\lambda_{ij-1}}{\beta_{j+1}\lambda_{ij+1}}$$
$$\times \cosh \lambda_{ij}(\psi_j - \psi_{j-1}) \cosh \lambda_{ij-1}(\psi_{j-1} - \psi_{j-2}). \quad (8)$$

Since $\psi_j > \psi_{j-1}$ from the configuration shown in ref. [1], then equations (5)–(8) show that all the elements of S are positive. Evaluating the first three matrix factors of equation (2) gives:

$$[B_t \quad 1]\begin{bmatrix} 1 & 0 \\ 0 & i\beta_J\lambda_{iJ} \end{bmatrix}\Omega[i\lambda_{iJ}(\psi_J - \psi_{J-1})] = [U \quad iV], \quad (9)$$

where

$$U = B_t \cosh \lambda_{iJ}(\psi_J - \psi_{J-1}) + \beta_J\lambda_{iJ} \times \sinh \lambda_{iJ}(\psi_J - \psi_{J-1}), \quad (10)$$

and

$$V = B_t \sinh \lambda_{iJ}(\psi_J - \psi_{J-1}) + \beta_J\lambda_{iJ} \times \cosh \lambda_{iJ}(\psi_J - \psi_{J-1}). \quad (11)$$

Evaluating the last three matrix factors in equation (2) gives:

$$\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\beta_1\lambda_{i1}}{\beta_2\lambda_{i2}} \end{bmatrix} \Omega[i\lambda_{i1}\psi_1]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cosh \lambda_{i1}\psi_1 \\ -i\dfrac{\beta_1\lambda_{i1}}{\beta_2\lambda_{i2}} \sinh \lambda_{i1}\psi_1 \end{bmatrix}. \quad (12)$$

Hence, evaluating the product of the three matrix factors at each end in equation (2) and also the remaining factors, leads to a matrix equation of the form:

$$[U \quad iV]\begin{bmatrix} T_{11} & iT_{12} \\ -iT_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \cosh \lambda_{i1}\psi_1 \\ -i\dfrac{\beta_1\lambda_{i1}}{\beta_2\lambda_{i2}} \sinh \lambda_{i1}\psi_1 \end{bmatrix} = 0. \quad (13)$$

Expressed in algebraic form, this equation becomes:

$$UT_{11} \cosh \lambda_{i1}\psi_1 + UT_{12}\frac{\beta_1\lambda_{i1}}{\beta_2\lambda_{i2}} \sinh \lambda_{i1}\psi_1 + VT_{21} \cosh \lambda_{i1}\psi_1 + VT_{22}\frac{\beta_1\lambda_{i1}}{\beta_2\lambda_{i2}} \sinh \lambda_{i1}\psi_1 = 0, \quad (14)$$

which is impossible since all the terms are positive. Hence the eigenvalues can never be imaginary in all the layers simultaneously.

When μ_n is zero, for which n is zero and there is no temperature variation along the slab, $\lambda_{j0m} = (1/\sqrt{\alpha_j})\lambda_{J0m}$ so that all the transverse eigenvalues in each layer are real, with the lowest value for each order 'm' occurring in the layer of highest diffusivity and the highest value occurring in the layer of lowest diffusivity; also, the value of the $(m+1)$th eigenvalue is higher than that of the mth eigenvalue in each layer. It is also clear from equation (23) in ref. [1] that the eigenvalue μ_n has a decreasing influence on the higher order transverse eigenvalues λ_{jnm}. Physically, this states that higher rates of change in temperature across a composite slab are less dependent than lower ones on temperature variations along the slab. Note that for a homogeneous slab, temperature variations in the two space dimensions are independent.

It was stated above that some transverse eigenvalues could be imaginary. It is shown in the Appendix that, for a two-layer slab, when the longitudinal eigenvalue is increased from μ_n to μ_{n+1}, then the value of any real transverse eigenvalue is decreased in the higher-diffusivity layer and increased in the lower-diffusivity layer. Strictly, the proof in the Appendix is only for a change Δ in μ which is sufficiently small for Δ^2 to be neglected and since from equation (11) in ref. [1] Δ has a value of π/γ, this implies a slab whose length is large relative to its thickness. However, the change in the value of the real transverse eigenvalues λ_{jnm} is always in the same direction, that is, always decreasing or increasing in the higher- or lower-diffusivity layers, respectively, for all values of m, and since the value of Δ for a thick slab is simply a multiple of the value of Δ for an arbitrarily thin slab, then the conclusion is true for all two-layer slabs. Since all the transverse eigenvalues must be real when n is zero, then they must remain real in the layer of lower diffusivity whereas in the layer of higher diffusivity, they must decrease from their values when 'n' is zero and they must all progressively become imaginary as 'n' is progressively increased. The lowest order transverse eigenvalue becomes imaginary firstly, then the second lowest order, etc.

If α_1 is greater than unity, then the transverse eigenvalues λ_1 in the lower layer of a two-layer slab become imaginary, or possibly zero, when from equation (23) in ref. [1]

$$\left(1 - \frac{1}{\alpha_1}\right)\mu_n^2 \geqslant \frac{1}{\alpha_1}\lambda_2^2.$$

If λ_1 is zero, then equation (51) in ref. [1] becomes

$$\lambda_2 \tan \lambda_2(\psi_2 - \psi_1) - B_t = 0, \quad (15)$$

which is of the same form as the transcendental equation (52) for the homogeneous slab. However, in the present case

$$\lambda_2 = (\alpha_1 - 1)^{1/2}\mu,$$

and equation (15) determines the value of 'μ' for a given value of B_t for which λ_1 is zero. Since 'μ' is dependent on the length of the slab, equation (15) defines the length for a particular type of slab and its environment for which there is at least one zero-valued eigenvalue in the lower layer.

If α_1 is less than unity, then the transverse eigenvalues λ_2 in the upper layer become imaginary, or possibly zero, when

$$(1-\alpha_1)\mu_n^2 \geqslant \alpha_1\lambda_1^2.$$

If λ_2 is zero, then equation (51) in ref. [1] becomes

$$B_t-[1+B_t(\psi_2-\psi_1)]\beta_1\lambda_1 \tan \lambda_1\psi_1 = 0, \quad (16)$$

where

$$\lambda_1 = \left(\frac{1-\alpha_1}{\alpha_1}\right)^{1/2}\mu,$$

and again equation (16) defines a length for a particular type of slab for which there is at least one zero-valued eigenvalue in the upper layer.

THE TRANSVERSE EIGENFUNCTIONS

The two-layer composite slab will be used to discuss the eigenfunctions associated with real, zero and imaginary eigenvalues and to deduce the related heat flows about the interface between the two layers. When the eigenvalues in both layers are real, then from equation (29) in ref. [1] and since $E_1 = 1$ and $G_1 = 0$ from equation (25) in ref. [1], the eigenfunctions in the lower layer have the form

$$\cos \lambda_1\psi,$$

and in the upper layer

$$\cos \lambda_2\psi+\frac{G_2}{E_2}\sin \lambda_2\psi,$$

where from equations (45) and (46) in ref. [1], with $\beta_2 = 1$

$$\frac{G_2}{E_2} = \frac{\lambda_2 \tan \lambda_2\psi_1-\beta_1\lambda_1 \tan \lambda_1\psi_1}{\lambda_2+\beta_1\lambda_1 \tan \lambda_2\psi_1 \tan \lambda_1\psi_1}. \quad (17)$$

If λ_1 is zero, then the form of the associated eigenfunction in the lower layer of the slab is a constant equal to unity, and in the upper layer is

$$\cos \lambda_2\psi+\tan \lambda_2\psi_1 \sin \lambda_2\psi.$$

If λ_2 is zero, then the form of the associated eigenfunction in the lower layer of the slab is

$$\cos \lambda_1\psi,$$

and in the upper layer is

$$1-\frac{\beta_1\lambda_1 \tan \lambda_1\psi_1}{1+\psi_1\beta_1\lambda_1 \tan \lambda_1\psi_1}\cdot\psi,$$

which is a line of constant slope.

It is interesting to note that when the zero-valued eigenvalue is in the lower layer, which must then be the more diffuse one, then no energy flows across the lower layer for that particular longitudinal and transverse temperature distribution. When the zero-valued eigenvalue is in the upper layer, which must then be the more diffuse one, there is no change in the stored energy at any point in that layer due to energy flowing across the layer and any such change must be due solely to energy flowing along that layer: again, this is only true for that particular longitudinal and transverse temperature distribution.

If λ_1 is imaginary and equal to $i\lambda_{i1}$, then the associated eigenfunction in the lower layer has the form

$$\cosh \lambda_{i1}\psi,$$

and in the upper layer

$$\cos \lambda_2\psi+\frac{\lambda_2 \tan \lambda_2\psi_1+\beta_1\lambda_{i1} \tanh \lambda_{i1}\psi_1}{\lambda_2-\beta_1\lambda_{i1} \tan \lambda_2\psi_1 \tanh \lambda_{i1}\psi_1}\cdot\sin \lambda_2\psi.$$

If λ_2 is imaginary and equal to $i\lambda_{i2}$, then the associated eigenfunction in the lower layer has the form

$$\cos \lambda_1\psi,$$

and in the upper layer

$$\cosh \lambda_{i2}\psi+\frac{\lambda_{i2} \tanh \lambda_{i2}\psi_1-\beta_1\lambda_1 \tan \lambda_1\psi_1}{\lambda_{i2}+\beta_1\lambda_1 \tanh \lambda_{i2}\psi_1 \tan \lambda_1\psi_1}\cdot\sinh \lambda_{i2}\psi.$$

The physical significance of the imaginary eigenvalues can be obtained by considering the shape of the associated eigenfunctions in the two layers. Figure 1(a) shows the eigenfunctions for a two-layer slab as λ_2 changes from being real, through zero, to being imaginary for consecutive values of μ_n and at the planes $\cos \mu_n\eta = \pm 1$; Fig. 1(b) shows the similar case for λ_1.

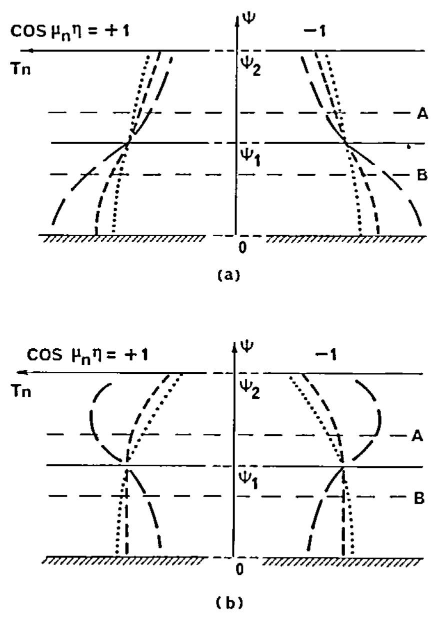

FIG. 1. Eigenfunctions for real and imaginary eigenvalues for a two-layer slab: (a) top layer more diffuse; (b) bottom layer more diffuse. ········, eigenvalue in more diffuse layer real; --------, eigenvalue in more diffuse layer zero; ———, eigenvalue in more diffuse layer imaginary.

Consider the planes A, B and cos $\mu_n\eta = \pm 1$ in Fig. 1(a). The longitudinal energy flow in both layers is away from the plane cos $\mu_n\eta = +1$ and towards the plane cos $\mu_n\eta = -1$. At the intersection of planes B and cos $\mu_n\eta = +1$, more energy is leaving than is arriving in the transverse direction and this excess must be from the stored energy here, whereas at the intersection of planes B and cos $\mu_n\eta = -1$, more energy is arriving than is leaving in the transverse direction and this excess energy must be stored here.

At the intersection of planes A and cos $\mu_n\eta = +1$:

λ_2 real: more energy is leaving than is arriving in the transverse direction and the excess must be from the stored energy here.

λ_2 zero: the energy leaving is equal to that arriving in the transverse direction and therefore the transverse energy flow has no effect on the temperature change here.

λ_2 imaginary: less energy is leaving than is arriving in the transverse direction, which cannot be associated with energy storage here since the energy is being dissipated through the top surface, and hence, this excess energy arriving in the transverse direction must be leaving in the longitudinal direction.

At the intersection of planes A and cos $\mu_n\eta = -1$:

λ_2 real: more energy is arriving than is leaving in the transverse direction and the excess must be stored here.

λ_2 zero: the energy leaving is equal to that arriving in the transverse direction and therefore the transverse energy flow has no effect on the temperature change here.

λ_2 imaginary: more energy is leaving than is arriving in the transverse direction, which cannot be associated with energy storage since the diffusion is into the slab from the surroundings, and hence, this excess energy leaving in the transverse direction must be arriving in the longitudinal direction.

An examination of Fig. 1(b) provides a similar argument to that for Fig. 1(a). Hence one can conclude that the imaginary eigenvalues are associated with the higher-diffusivity layer effectively short-circuiting the lower-diffusivity layer in the longitudinal direction, for those particular transverse and longitudinal temperature distributions.

It is interesting to note that when B_t is zero, which it is for a fully insulated slab, then when μ is zero, a possible solution of equation (51) in ref. [1] is that both λ_1 and λ_2 are equal to zero. Hence, when μ is not zero, the lowest-order transverse eigenvalue must be imaginary in the layer of higher diffusivity, and real and greater than zero in the layer of lower diffusivity. In physical terms, this tells us that while it is possible to have a temperature variation across a fully insulated composite slab with no temperature variation along it, it is impossible to have temperature variation along the slab without having temperature variation across it.

CONCLUSIONS

It has been shown that the transverse eigenvalues decrease in magnitude in the more diffusive layer of a two-dimensional two-layer composite slab as the longitudinal eigenvalues increase, and conversely, they increase in the less diffusive layer. In the homogeneous slab, the transverse eigenvalues are independent of the longitudinal eigenvalues and the solution for the composite slab therefore shows that the rate of decay of temperature variations along the slab is decreased in the higher-diffusivity layer by the presence of the lower-diffusivity layer, but it is enhanced in the lower-diffusivity layer by the presence of the higher-diffusivity layer.

The transverse eigenvalues in the higher-diffusivity layer progressively become imaginary as the order of the longitudinal eigenfunction is increased. These imaginary eigenvalues reflect that for that order of longitudinal and transverse temperature distribution, energy is being transferred between different parts of the lower-diffusivity layer via the higher-diffusivity layer. The sum of the squares of the real longitudinal eigenvalue and the imaginary transverse eigenvalue is always positive because the sum is related to the sum of the squares of the eigenvalues in the other layer, and the eigenvalues in the layer of lower diffusivity are always real; hence, the complete eigenfunction is decaying in both layers. The solution also shows that steeper temperature gradients across the lower-diffusivity layer, that is higher values of λ_{jnm}, require steeper temperature gradients along the slab, that is higher values of μ_n for energy to flow between different parts of the lower-diffusivity layer via the higher-diffusivity layer.

For the special case of a fully insulated composite slab, it is possible to have temperature variations across the slab with none along it, but the converse is impossible.

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APPENDIX

Change in the transverse eigenvalues for a change in the longitudinal eigenvalue for a two-layer slab

The transcendental equation governing the eigenvalues is equation (51) in ref. [1], which can be written as

$$\lambda_2^2 \tan \lambda_2(\psi_2-\psi_1)+\beta_1\lambda_1\lambda_2 \tan \lambda_1\psi_1+B_t\beta_1\lambda_1 \times \tan \lambda_1\psi_1 \tan \lambda_2(\psi_2-\psi_1)-B_t\lambda_2 = 0, \quad \text{(A1)}$$

where from equation (23) in ref. [1]

$$\lambda_2^2+\mu^2 = \alpha_1(\lambda_1^2+\mu^2). \quad \text{(A2)}$$

If the nth longitudinal eigenvalue is μ and the $(n+1)$th is '$\mu+\Delta$', and if the corresponding transverse eigenvalues are λ_2 together with λ_1, and $\lambda_2+\Delta_2$ together with $\lambda_1+\Delta_1$

respectively, then

$$(\lambda_2+\Delta_2)^2+(\mu+\Delta)^2 = \alpha_1[(\lambda_1+\Delta_1)^2+(\mu+\Delta)^2]. \qquad \text{(A3)}$$

If Δ, Δ_1 and Δ_2 are all sufficiently small that their squares and higher powers may be neglected, then equation (A3) can be reduced to

$$\lambda_2\Delta_2+\mu\Delta = \alpha_1(\lambda_1\Delta_1+\mu\Delta). \qquad \text{(A4)}$$

Also, from equation (A1)

$$(\lambda_2+\Delta_2)^2 \tan(\lambda_2+\Delta_2)+\beta_1(\lambda_1+\Delta_1)(\lambda_2+\Delta_2) \times \tan\{(\lambda_1+\Delta_1)\psi_1\}+B_t\beta_1(\lambda_1+\Delta_1)\tan\{(\lambda_1+\Delta_1)\psi_1\} \times \tan\{(\lambda_2+\Delta_2)(\psi_2-\psi_1)\}-B_t(\lambda_2+\Delta_2) = 0. \qquad \text{(A5)}$$

Expanding the terms and neglecting the second and higher power terms in Δ_1 and Δ_2, which allows $\tan\Delta_j$ to be replaced by Δ_j, then equation (A5) can be reduced to

$$\lambda_2\Delta_2\left\{\begin{array}{l}\left[\lambda_2(\psi_2-\psi_1)+2\tan\lambda_2(\psi_2-\psi_1)+\beta_1\dfrac{\lambda_1}{\lambda_2}\tan\lambda_1\psi_1\right.\\ \quad -\beta_1\lambda_1(\psi_2-\psi_1)\tan\lambda_1\psi_1\tan\lambda_2(\psi_2-\psi_1)\\ \quad +B_t(\psi_2-\psi_1)\beta_1\dfrac{\lambda_1}{\lambda_2}\tan\lambda_1\psi_1+B_t(\psi_2-\psi_1)\\ \quad \left.\times\tan\lambda_2(\psi_2-\psi_1)-\dfrac{B_t}{\lambda_2}\right]\end{array}\right.$$

$$+\lambda_1\Delta_1\left\{\begin{array}{l}\left[-\dfrac{\lambda_2^2}{\lambda_1}\psi_1\tan\lambda_1\psi_1\tan\lambda_2(\psi_2-\psi_1)+\beta_1\lambda_2\psi_1\right.\\ \quad +\beta_1\dfrac{\lambda_2}{\lambda_1}\tan\lambda_1\psi_1+B_t\dfrac{\beta_1}{\lambda_1}\tan\lambda_1\psi_1\tan\lambda_2(\psi_2-\psi_1)\\ \quad \left.+B_t\beta_1\psi_1\tan\lambda_2(\psi_2-\psi_1)+B_t\dfrac{\lambda_2}{\lambda_1}\psi_1\tan\lambda_1\psi_1\right]\end{array}\right.$$

$$= 0. \qquad \text{(A6)}$$

Substituting for $\lambda_2\Delta_2$ from equation (A4) into equation (A5), and using equation (A1) to simplify the resulting equation gives

$$\lambda_1\Delta_1\left\{\alpha_1+\frac{\lambda_2}{\lambda_1} \times\frac{\beta_1[\lambda_2+B_t\tan\lambda_2(\psi_2-\psi_1)]\lambda_1\psi_1Z_1}{\lambda_2^2(\psi_2-\psi_1)Z_2+B_t(\psi_2-\psi_1)\beta_1\lambda_1\tan\lambda_1\psi_1\cdot Z_3}\right\} +(\alpha_1-1)\mu\Delta = 0, \qquad \text{(A7)}$$

which, by squaring the denominator in the factor multiplying λ_2/λ_1, leads to

$$\lambda_1\Delta_1\left\{\alpha_1+\frac{\lambda_2^2\psi_1\beta_1Z_1(\lambda_2^2Z_2+B_t^2Z_3+2\lambda_2B_tZ_4)}{(\lambda_2^2Z_2+B_t\beta_1\lambda_1\tan\lambda_1\psi_1\cdot Z_3)^2}\right\} +(\alpha_1-1)\mu\Delta = 0, \qquad \text{(A8)}$$

where

$$Z_1 = 1+\tan^2\lambda_1\psi_1+\frac{\tan\lambda_1\psi_1}{\lambda_1\psi_1}, \qquad \text{(A9)}$$

$$Z_2 = 1+\tan^2\lambda_2(\psi_2-\psi_1)+\frac{\tan\lambda_2(\psi_2-\psi_1)}{\lambda_2(\psi_2-\psi_1)}, \qquad \text{(A10)}$$

$$Z_3 = 1+\tan^2\lambda_2(\psi_2-\psi_1)-\frac{\tan\lambda_2(\psi_2-\psi_1)}{\lambda_2(\psi_2-\psi_1)}, \qquad \text{(A11)}$$

and

$$Z_4 = \frac{\tan^2\lambda_2(\psi_2-\psi_1)}{\lambda_2(\psi_2-\psi_1)}. \qquad \text{(A12)}$$

Note that Z_1–Z_4 are always positive for all real values of λ_1 and λ_2. Therefore equation (A8) shows that if $\alpha_1 < 1$, then Δ_1 is positive when Δ is positive, and if $\alpha_1 > 1$, then Δ_1 is negative when Δ is positive.

Combining equations (A8) and (A4) gives

$$\lambda_2\Delta_2 = (\alpha_1-1)\mu\Delta\{\lambda_2^2\psi_1\beta_1Z_1(\lambda_2^2Z_2+B_t^2Z_3+2\lambda_2B_tZ_4) \div [(\lambda_2^2Z_2+B_t\beta_1\lambda_1\tan\lambda_1\psi_1\cdot Z_3)^2+\lambda_2^2\psi_1\beta_1Z_1 \times(\lambda_2^2Z_2+B_t^2Z_3+2\lambda_2B_tZ_4)]\}, \qquad \text{(A13)}$$

which shows that if $\alpha_1 < 1$, then Δ_2 is negative when Δ is positive, and if $\alpha_1 > 1$, then Δ_2 is positive when Δ is positive.

Hence, for $\alpha_1 < 1$, λ_1 increases and λ_2 decreases as μ increases, and for $\alpha_1 > 1$, λ_1 decreases and λ_2 increases as μ increases.

CONDUCTION VARIABLE DANS UNE PLAQUE COMPOSITE BIDIMENSIONNELLE. INTERPRETATION PHYSIQUE DES MODES DE TEMPERATURE

Résumé—On montre que les valeurs propres des modes de température à travers une plaque composite à deux couches peuvent devenir progressivement imaginaires dans la couche la plus diffuse, pour les modes de température d'orde élevé le long de la plaque. Un argument logique est présenté pour montrer que cela représente le fait que la couche la plus diffusante court-circuite la moins diffusante en dissipant les variations de température suivant la longueur. On montre aussi que pour une plaque composite totalement calorifugée, des variations de température longitudinales sont accompagnées par des variations de température transversales.

INSTATIONÄRE WÄRMELEITUNG IN EINER ZWEIDIMENSIONALEN GESCHICHTETEN PLATTE—II. PHYSIKALISCHE DEUTUNG DER LÖSUNGEN FÜR DIE TEMPERATUR

Zusammenfassung—Es wird gezeigt, daß die Eigenwerte der Lösungen des Temperaturverlaufs quer zu einer zweischichtigen Platte in der besser leitenden Schicht für die Teillösungen höherer Ordnung in Längsrichtung zunehmend imaginär werden müssen. Dieser Umstand läßt sich so deuten, daß die besser wärmeleitende Schicht die schlechter leitende hinsichtlich des Ausgleichs von Temperaturunterschieden in Längsrichtung kurzschließt. Weiterhin wird gezeigt, daß für eine vollkommen isolierte, geschichtete Platte Temperaturunterschiede in Längsrichtung immer von Temperaturunterschieden in Querrichtung begleitet sein müssen.

НЕУСТАНОВИВШАЯСЯ ПЕРЕДАЧА ТЕПЛА ТЕПЛОПРОВОДНОСТЬЮ В ДВУМЕРНОЙ КОМПОЗИТНОЙ ПЛИТЕ—II. ФИЗИЧЕСКОЕ ОБОСНОВАНИЕ ТЕМПЕРАТУРНЫХ МОД

Аннотация—Показано, что собственные температурные моды, распространяющиеся поперек двухслойной композитной плиты, должны постепенно переходить в затухающие в диффузном слое для температурных мод высшего порядка, распространяющихся вдоль плиты. Приведено логическое доказательство того, что чем более эффективно закорочен диффузный слой, тем он меньше при продольных диссипативных изменениях температуры. Показано также, что для полностью изолированной композитной плиты продольные изменения температуры вдоль плиты должны сопровождаться поперечными температурными изменениями.